FRAC TURE MECHANICS
APPROACH TO HY DRAULIC
FRAC TURING STRESS
MEASUREMENTS

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6.1. INTRODUCTION

In rock mechanics the term hydraulic fracturing is used for fluid injection operations in sealed-off borehole intervals to induce and propagate tensile fractures. In the oil industry it was first applied, in 1948, to stimulate productivity from low permeability oil-bearing formations (Clark, 1949). Since then, hydraulic fracturing has been used for deep gas or water well stimulation, for geothermal energy extraction to induce heat exchange surfaces, in coal gasification pilot projects, and for in situ stress measurements.

The classical treatment of hydraulic fracturing is based on Kirsch’s (1898) solution for the stress distribution around a circular hole in a homogeneous, isotropic, elastic material subjected to external compression. It was used by Hubbert and Willis (1957) who stated that a fracture in the borehole wall will be initiated if the acting fluid pressure in a hole exceeds the minimum
tangential stress and the tensile strength of the material. For fracturing in vertical holes drilled from the surface this is generally expressed by the relation

\[ P_c = 3S_h - S_H + P_{co} - P_o, \]  

(6.1)

where the critical pressure at fracture initiation, \( P_c \), is denoted as breakdown pressure. \( S_H \) and \( S_h \) are the principal horizontal far field stresses, \( P_{co} \) is the tensile strength of the rock and \( P_o \) is the pore pressure. Since one also assumes that the fracture propagates in the direction of least resistance, the pressure to merely keep an induced vertical fracture open is equal to the least principal horizontal stress \( S_h \):

\[ P_{st} = S_h. \]  

(6.2)

In practice \( P_{st} \) is called the shut-in pressure. For the specific case \( S_H = S_h = S \) Equation (6.1) reduces to

\[ P_c = 2S + P_{co} - P_o \]  

(6.3)

which predicts fracture initiation in crustal environments characterized by a lithostatic stress field (\( S = S_V = \rho_s g z \) overburden pressure) or in internally pressurized thick cylinders subjected to a confining pressure \( S = p_m \).

This classical approach neglects the fact that real materials such as rocks contain pre-existing fractures, into which pressurizing fluid can penetrate and contribute to the stress intensity at the crack tips prior to fracturing. The problem of hydraulic fracturing, therefore, reduces to defining the critical condition for the growth of existing cracks, rather than predicting crack initiation within idealized materials.

During the last two decades, the problem was attacked by numerous fracture mechanics models, which also take into account fluid flow within the propagating fracture, fluid losses into the rock due to permeability, as well as boundary conditions such as stress gradients or material property changes at formation boundaries. A closed three-dimensional solution for the general complex problem is not yet available. This chapter gives some examples of laboratory and field hydraulic fracturing measurements and demonstrates the application of fracture mechanics to hydraulic fracturing problems.

### 6.2. EXPERIMENTAL OBSERVATIONS

#### 6.2.1. Laboratory Experiments

Laboratory hydraulic fracturing tests are performed to supply material properties for hydrofracture field data interpretation (e.g. \( P_{co} \) in Equation (6.1)), or to study the process of fracture growth under given environmental conditions. Tests are either carried out on cylindrical core
Figure 6.1. Breakdown pressure $p_c$ as a function of confining pressure $p_m$ for triaxially loaded and internally pressurized mini-cores of Falkenberg granite. 10 bar = 1 MPa.

samples subjected to confining pressures $p_m$, or on rectangular blocks loaded triaxially by $S_h \neq S_h \neq S_v$. The fluid pressure is applied to sealed-off boreholes which are generally drilled parallel to $S_v$.

Typical results from fracturing tests on granite minicores are given in Fig. 6.1. The samples were 3 cm in diameter and contained an axial drillhole with a diameter of 2.5 mm. The confining pressures ranged up to 800 bar. Hydraulic oil with a viscosity of 32 eSt ($32 \times 10^{-6}$ m$^2$ s$^{-1}$) was used as fracture fluid injected at a pumping rate of 5 bar s$^{-1}$. In all cases symmetrical axial fractures propagated throughout the samples. The relationship between the breakdown pressure $p_c$ at unstable crack growth and the applied confining pressure $p_m$ can be expressed as

$$p_c = kp_m + p_{\infty} \quad (6.4)$$

with $k = 1.04 \pm 0.04$ and $p_{\infty} = 166 \pm 15$ bar. Similar values are observed for other rock types when tested under similar conditions (Table 6.1). The relation corresponds to Equation (6.3) except that the empirical fracture coefficients $k$ are considerably smaller than expected from the stress concentration at the borehole wall. They may eventually approach the theoretical value $k = 2$ if high viscosity fracture fluids or high pumping rates are used. This suggests that fluid penetration into pre-existing microcracks prior to unstable crack growth plays an important role.

Neglecting at present the effect of specimen size (see below), the laboratory fracture coefficients may be used to estimate breakdown pressures, $P_c$. 


Table 6.1
Hydrofracture tensile strength $p_{co}$ and fracture coefficient $k$ (Equation (6.4)) of minicore from various rocks. $p_m$ is the confining pressure. 10 bar = 1 MPa.

<table>
<thead>
<tr>
<th>Rock type</th>
<th>Number of specimens</th>
<th>Confining pressure range (bar)</th>
<th>$p_{co}$ (bar)</th>
<th>$k$</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>granite, Falkenberg</td>
<td>32</td>
<td>0–800</td>
<td>166 ± 15</td>
<td>1.04 ± 0.04</td>
<td>Müller and Rummel, 1987</td>
</tr>
<tr>
<td>gabbro, Kolar Gold Field, India</td>
<td>20</td>
<td>0–500</td>
<td>183 ± 24</td>
<td>1.05 ± 0.09</td>
<td>Gowd et al., 1983</td>
</tr>
<tr>
<td>sandstone, Ruhr</td>
<td>46</td>
<td>0–600</td>
<td>266 ± 34</td>
<td>1.49 ± 0.15</td>
<td>Winter, 1983</td>
</tr>
<tr>
<td>sandstone, Venn massif</td>
<td>11</td>
<td>0–300</td>
<td>128 ± 41</td>
<td>1.35 ± 0.26</td>
<td>Rummel and Baumgartner, 1985</td>
</tr>
<tr>
<td>sandstone, bunt-sandstone</td>
<td>9</td>
<td>0–70</td>
<td>132 ± 7</td>
<td>1.23 ± 0.17</td>
<td>Rummel, int. rep.</td>
</tr>
<tr>
<td>limestone</td>
<td>12</td>
<td>0–185</td>
<td>227 ± 26</td>
<td>0.95 ± 0.25</td>
<td>Rummel, int. rep.</td>
</tr>
<tr>
<td>marble, Carrara</td>
<td>7</td>
<td>0–245</td>
<td>194 ± 5</td>
<td>1.26 ± 0.04</td>
<td>Möhring-Erdman and Rummel, 1987</td>
</tr>
<tr>
<td>rocksalt, Asse</td>
<td>45</td>
<td>0–300</td>
<td>30 ± 4</td>
<td>1.20 ± 0.03</td>
<td>Rummel, int. rep.</td>
</tr>
</tbody>
</table>
for in situ fracture operations, particularly for crustal regimes where we expect a state of stress close to lithostatic. Thus, the breakdown pressure \( P_e \) is given by

\[
P_e = k^* z + P_{\infty}
\]

with \( k^* = g(k_0 - \varrho_n) \). Taking typical laboratory data for \( k \) and the densities of rock and the borehole fluid (\( k = 1.04, \varrho_r = 2.65 \text{ g cm}^{-3}, \varrho_n = 1 \text{ g cm}^{-3} \)) we obtain a fracture gradient of \( k^* = 0.172 \text{ bar m}^{-1} \). This value appears to be in good agreement with fracture gradients reported from deep oil and gas well stimulation work (e.g. \( k^{**} = 0.188 \), Klose and Krömer, 1983) although such fracture gradients are defined slightly differently (\( k^{**} = P_e^* / z \)) and injection rates and fracture fluid viscosities are significantly different to laboratory conditions.\(^1\) As discussed below, the comparison is invalid if all experimental parameters are not included in the evaluation.

Similarly, the results of hydraulic fracturing tests on triaxially loaded blocks can be described by a relation corresponding to Equation (6.1), but replacing the theoretical stress concentration factors (3 and -1) by fracture coefficients \( k_1 \) and \( k_2 \) that need to be determined

\[
p_r = (k_1 \kappa + k_2) S_H
\]

where \( \kappa = S_h / S_H \) and \( p_r = p_e - P_{\infty} \) which denote the refraction pressure necessary to reopen a previously induced fracture. A typical pressure record for a test conducted on a 1 m\(^3\) granite block is given in Fig. 6.2. The block was

\(^1\) It should also be noted that fracture gradients in the oil industry generally are defined with respect to the shut-in pressure \( P_{\text{sh}} \), i.e. \( k^{***} = P_{\text{sh}}^* / z \); \( P_{\text{sh}}^* \) and \( P_e^* \) surface pressures.
excavated on the basement of a granite quarry and was loaded biaxially by flat-jacks with $S_H = 50$ bars and $S_h = 20$ bars. Fracture initiation occurred at $p_c = 155$ bars and the shut-in pressure $p_{si}$ corresponded to $S_h$. Although the re-opening pressure $P_r$ is difficult to identify from the pressure record, $P_r$ can be estimated as 15 to 25 bars which then yields a hydrofracture tensile strength for the granite block of $p_{co} \approx 135$ bars.

The fact that unstable fracture growth occurs in the form of discrete events is illustrated in Fig. 6.3, which shows the fracture growth pattern in a plexiglas cube ($15 \times 15 \times 15$ cm) loaded biaxially in a triaxial press. Initially, a given axial crack was propagated for some distance. Then the applied external horizontal stresses were rotated by $\pi/2$, which caused the fracture
also to change its direction. The subsequent crack growth pattern is marked by circular lines which characterize step-phases of crack growth whenever the excess potential energy of the hydraulic loading system was consumed in the creation of new fracture surface. The evaluation of energy supply and the increase of fracture surface during each event yields a specific surface energy value for plexiglas of $\gamma = 150 \text{ J m}^{-2}$. This value is in good agreement with data derived from fracture toughness tests (Winter, 1983).

### 6.2.2. Field Experiments

In order to set limits for the present article it is useful to distinguish between massive hydraulic fracturing operations (MHF) as used in oil and gas well stimulation or recently also in geothermal projects, and mini- or micro-hydrofracturing as applied to induce an initial fracture for MHF, to test the fracture resistance of formations, or to measure in situ crustal stresses. During the largest European MHG treatment in the well Soehlingen Z 4, FRG, a total of 2582 m$^3$ of fracture fluid and 546 tons of high strength proppant were injected at an average rate of 6.4 m$^3 \text{ min}^{-1}$ (Klose and Krömer, 1983). The induced fracture area is estimated between $9.2 \times 10^4$ m$^2$ (theoretical design) and $3.4 \times 10^4$ m$^2$ (effectively propped). Similarly, effective heat transfer areas induced by MHF for hot dry rock geothermal energy extraction projects (Los Alamos and Cornwall) are in the order of $10^5$ m$^2$. The ultimate goal of such HDR projects is to achieve an effective fracture heat exchange area of $2 \times 10^6$ m$^2$.

In comparison, mini- or microfracture operations use pumping rates of 1 m$^3 \text{ min}^{-1}$ to several litres per minute, and intend to achieve fracture areas with an extension of only a few metres. At the Falkenberg geothermal test site, Falkenberg granite massif, NE Bavaria, a planar structure was induced at a depth of 250 m using a maximum pumping rate of 200 l min$^{-1}$. The present fracture area is estimated as $2 \times 10^4$ m$^2$ and can take a fluid volume of 20 m$^3$ (Rummel and Kappelmeyer, 1982). The pressure records during the initial growth of this fracture are given in Fig. 6.4. This figure also shows the occurrence of seismic events which indicate periodic unstable fracture growth whenever the criterion for fracture propagation is reached.

For in situ hydraulic fracturing stress measurements only a total of 10 to 50 l of fracture fluid are injected into the induced fracture at pumping rates of a few litres per minute. This can be achieved by a wireline hydrofracture system as shown in Fig. 6.5, where the fracture fluid is pumped into the scaled-off borehole section via a high pressure rubber hose. The system enables numerous tests to be performed in a reasonable time and so allows the possibility of obtaining stress logs along continuous depth profiles. It also eliminates the presence of drill rigs and large pumping units. The method is described in detail by Rummel et al. (1983), Haimson and Lee (1984) and
Figure 6.4. Pressure-time record during the initiation of a large hydrofracture at the Falkenberg geothermal test site. Depth 250 m, pumping rate 200 l min⁻¹, ◦ indicate the occurrence of seismic events during fracture growth. P₁: refracture pressure, I: injected volume 0.2 m³, II: 1 m³, III: 5 m³. 10 bar = 1 MPa.

Figure 6.5. The Bochum Wireline Hydrofracture System for stress measurements. 1, data recording; 2, pressure and flow control; 3, pumping unit; 4, hydraulic winch; 5, depth measurement system; 6, cable drum; 7, high pressure hose; 8, seven conductor cable; 9, clamp to connect hose to cable; 10, cable head with pressure transducer; 11, pressure release valve; 12, push-pull valve; 13, packer elements; 14, injection interval; 15, signal lead through for additional components.
Figure 6.6. A typical pressure-time record for a micro-hydrofracture test for stress measurements in a 100 m diameter borehole at the Falkenberg granite test site. P, F, RF and SP indicate various phases of the test to measure in situ permeability (P), to induce the fracture (F), to enlarge the fracture (RF) and to reopen the fracture slowly (SL). P<sub>c</sub>, P<sub>r</sub>, P<sub>s</sub> and P<sub>r</sub> are critical pressures at fracture initiation, reopening and after shut-in and during pumping. Depth of test 210 m, Q pumping rate: 10 bar = 1 MPa.

Baumgärtner et al. (1987). A typical pressure record during a microfracture test in one of the 300 m deep Falkenberg granite boreholes is shown in Fig. 6.6. The record is quite similar to the one for the large scale test (Fig. 6.4) except that the shut-in pressure is less distinct. This may be explained if the small fluid volume injected does not open the fracture sufficiently. A summary of most of the microfracture tests in the Falkenberg drillholes is given in Fig. 6.7 which shows both the refracture pressure P<sub>r</sub> and the shut-in pressure P<sub>s</sub> as a function of depth z. The scatter of the data is quite considerable due to the fact that the directions of the induced fractures generally deviate from the ideal direction expected in the classical approach (vertical fractures oriented parallel to S<sub>H</sub> for stress situations defined by S<sub>V</sub> > S<sub>H</sub> or S<sub>H</sub> > S<sub>V</sub> > S<sub>H</sub>, respectively). This is particularly true for crystalline rock formations which contain pre-existing microfractures with preferred orientations. By taking into account the azimuth and dip of each fracture as observed from impression packer tests or from acoustic televiewer tests, we may generalize Equations (6.1) and (6.2) into

\[
P_{r,i} = \rho g z_i \cos^2 \alpha_i + \sin^2 \alpha_i \left( S_{H,o} + S_{k,o} + (\delta_H + \delta_n)z_i - 2[S_{H,o} - S_{k,o} + (\delta_H - \delta_n)z_i] \cos 2(\theta_i - \theta^*) \right) - P_o \tag{6.7}
\]

\[
P_{s,i} = \rho g z_i \cos^2 \alpha_i + \frac{1}{2} \sin^2 \alpha_i \left( S_{H,o} + S_{k,o} + (\delta_H + \delta_n)z_i - [S_{H,o} - S_{k,o} - (\delta_H - \delta_n)z_i] \cos 2(\theta_i - \theta^*) \right) \tag{6.8}
\]

where \( S_{V,i} = \rho g z_i \) is the overburden stress at a depth \( z_i \), \( \alpha_i \) is the dip angle of the fracture plane with respect to the horizontal, \( \theta_i \) is the strike of the fracture plane with respect to magnetic north, \( \theta^* \) is the direction of \( S_{H,o} \) with
respect to north. \( S_{H,0}, S_{h,0}, \delta_{H}, \delta_{h} \) are eigenvalues of the stress matrix \( \mathbf{S}(z) = \mathbf{S}_0 + \delta_0 \), i.e. \( S_{H,0} \) and \( S_{h,0} \) are the horizontal stresses at the surface, and \( \delta_{H} \) and \( \delta_{h} \) are the stress gradients of \( S_{H} \) and \( S_{h} \) with respect to depth, and \( P_0 \) is the pore pressure at a depth \( z_i \).

Equations (6.7) and (6.8) reduce to Equations (6.1) and (6.2) for vertical fractures (\( \alpha_i = \pi/2 \)) which are aligned with the direction of \( \mathbf{S}_0(\theta_i = \theta^*) \). Since the exact identification of \( P_i \) values is often problematic in the evaluation of the pressure record, and \( P_i \) values may also be influenced by imposed stresses from the packers, in practice the derivation of the principal stresses is carried out on the basis of \( P_{si} \) data only by using Equation (6.8) which includes five unknowns \( (S_{H,0}, S_{h,0}, \delta_{H}, \delta_{h}, \theta^*) \). Using a random sample technique theoretical \( P_{si} \) values can be calculated and compared with the observed \( P_{si} \) data provided sufficient field data are available. The procedure suggested by Baumgärtner et al. (1987) is as follows

(i) definition of sample ranges for the five unknowns;
(ii) selection of sets of five variables and calculation of the corresponding \( P_{si} \) values for various depth positions \( z_i \);
(iii) determination of the parameter set with least sum of squared errors as the best solution;
(iv) calculation of the principal stresses \( S_{H} \) and \( S_{h} \) by the relations

\[
S_{H} = \frac{1}{2} [S_{H,0} + S_{h,0} + (\delta_{H} + \delta_{h})z + \Delta] \quad (6.9)
\]
\[
S_{h} = \frac{1}{2} [S_{H,0} + S_{h,0} + (\delta_{H} + \delta_{h})z - \Delta] \quad (6.10)
\]
\[
\Delta = S_{H,0} - S_{h,0} + (\delta_{H} - \delta_{h})z. \quad (6.11)
\]

The general solution of the problem (Baumgärtner et al., 1987) also considers the case that the principal stresses may rotate with depth. The
advantage of this procedure to derive crustal stresses from hydraulic fracturing data is mainly that no narrow a priori assumptions are necessary. In practice, however, a clue to the regional stress situation will confine the sample range and lead to better solutions due to high sample density.

The application of this procedure to the Falkenberg shut-in pressure data given in Fig. 6.7 resulted in a calculated stress state (Fig. 6.8) which seems consistent with the regional tectonics of this particular area. The Falkenberg granite massif is part of the Bohemian block with no evidence of recent tectonic activity. The calculation considers about $6 \times 10^5$ parameter sets within reasonable boundary conditions and suggests that the direction of maximum crustal compression is N120°E (Fig. 6.9).
6.3. FRACTURE MECHANICS APPROACH TO HYDRAULIC FRACTURING

In modelling initial hydraulic fracture growth in rocks we assume a symmetrical double crack of half length $a$, extending from a circular hole in an otherwise intact infinite plate subjected to compressive far field stresses $S_H$ and $S_h$ (Fig. 6.10). For simplicity, it is supposed at present that the crack normal is parallel to the direction of $S_h$. Fluid pressure is applied to the wall of the hole and the pressurized fluid may also penetrate into the crack. In spite of this complex stress system the intensity of the stress field in the vicinity of the crack tips can easily be formulated using the principle of superposition of stress intensity factors from each loading source (Fig. 6.11):

$$K_1(S_H, S_h, p, p_a) = K_1(S_H) + K_1(S_h) + K_1(p) + K_1(p_a)$$  \hspace{1cm} (6.12)

where the $K_1$ indicate stress intensity factors for mode I crack propagation, $p$ is the fluid pressure acting on the wall of the hole, and $p_a = p_a(x, 0)$ characterizes the fluid pressure distribution along the crack from $x = (R, -R)$ to $x = (R + a, -R - a)$.

The two-dimensional problem as outlined above was investigated by several researchers employing various analytical and numerical methods.

Figure 6.10. Fracture mechanics model for growth of a microcrack with half-length $a$, emanating from a pressurized borehole in an infinite plate subjected to the far field stresses $S_H$ and $S_h$. 
The most comprehensive contributions appear to be those due to Bowie (1956), Wigglesworth (1958) and Newman (1969, 1971). The problem was further discussed by Bueckner (1960), Hardy (1973) and Ingraffea (1977). Here we follow a purely analytical solution suggested by Rummel and Winter (1982, 1983) and presented in a comprehensive form by Winter (1983). This simple approach is based upon the general formulation of the stress intensity factor for a tension crack of half length \( a \) in an infinite plate given by Paris and Sih (1965) as

\[
K_I = (\pi a)^{1/2} \int_{-a}^{a} \sigma_y(x,0) \left( \frac{a + x}{a - x} \right)^{1/2} \, dx
\]

(6.13)

where \( \sigma_y \) is the tangential stress in the crack plane \( y = 0 \) (Fig. 6.10). The solution consists of the determination of the tangential stress functions \( \sigma_y(x,0) \) for each of the load sources and the integration of Equation (6.13) for each case.

6.3.1. Derivation of \( K_I (S_h) \)

Neglecting the existence of a crack the tangential stress \( \sigma_\theta \) around a circular hole of radius \( R \) in an infinite plate subjected to the far field stresses \( S_H \) and \( S_h \) is given by Kirsch (1898) as

\[
\sigma_\theta(r,\theta) = \frac{1}{2}(S_H + S_h) \left[ 1 + \left( \frac{R}{r} \right)^2 \right] - \frac{1}{2}(S_H - S_h) \left[ 1 + 3 \left( \frac{R}{r} \right)^4 \right] \cos 2\theta.
\]

(6.14)

Then the tangential stress for \( S_h = 0 \) in the \( x \) direction is given by
Substituting the relation into Equation (6.13) leads to

$$K_I(S_H) = \frac{S_H}{2[a(R + a)]^{1/2}} \int_{-(R + a)}^{R + a} \left[ \left( \frac{R}{x} \right)^2 - 3 \left( \frac{R}{x} \right)^4 \right] \left( \frac{R + a + x}{R + a - x} \right)^{1/2} dx$$

and by integration to

$$K_I(S_H) = -2S_H \sqrt{R} \left( \frac{b^2 - 1}{\pi b^7} \right)^{1/2}$$

with $b = 1 + a/R$. It should be noted that the hole was neglected for the integration. The integration was limited to the intervals $(-(R + a), R)$ and $(R + a, R)$. Further, compressive stresses are taken as positive which implies a change of sign in Equation (6.17) to take into account the fact that $K_I$ is positive for crack opening. Thus,

$$K_I(S_H) = -S_H \sqrt{R} f(b)$$

$$f(b) = -2[(b^2 - 1)/\pi b^7]^{1/2}.$$ 

The dimensionless stress intensity function $f$ is shown in Fig. 6.12 in comparison with the solutions given by Bowie (1956) and Newman (1971), which were obtained on the basis of a complex variable method using a.

Figure 6.12. Dimensionless stress intensity functions $f(1 + a/R)$ for a crack of half-length $a$ originating at the boundary of a circular hole in an infinite plate subjected to uniaxial compression parallel to the crack. 1, Equation (6.19); 2, Bowie (1956), Newman (1971). Normalized crack length $b = 1 + (a/R)$. 
conformal mapping procedure and a numerical boundary collocation technique, respectively.

6.3.2. Derivation of $K_1(S_h)$

Using the Kirsch equation (Equation (6.14)) for $S_{11} = 0$ yields a tangential stress of

$$\sigma_y(x,0) = \frac{1}{2} S_h \left[ 2 + \left( \frac{R}{x} \right)^2 + 3 \left( \frac{R}{x} \right)^4 \right].$$

Substitution into Equation (6.13) and integration over the intervals $\{-(R + a), -R\}, \{R + a, R\}$ leads to

$$K_1(S_h) = S_h \sqrt{R} \left[ (\pi b)^{1/2} \left( 1 - \frac{2}{\pi} \sin^{-1} \frac{1}{b} \right) + 2(b^2 + 1) \left( \frac{(b^2 - 1)}{\pi b^4} \right)^{1/2} \right]$$

or with consideration of the sign convention

$$K_1(S_h) = -S_h \sqrt{R} g(b)$$

$$g(b) = (\pi b)^{1/2} \left( 1 - \frac{2}{\pi} \sin^{-1} \frac{1}{b} \right) + 2(b^2 + 1) \left( \frac{(b^2 - 1)}{\pi b^4} \right)^{1/2}$$

The dimensionless stress intensity function $g$ is shown in Fig. 6.13 in comparison with the solution presented by Bowie (1956) and Newman (1969).

![Figure 6.13](image-url)
6.3.3. Derivation of $K_1(p)$ and $K_1(p_a)$

The stress intensity function due to fluid pressurization depends both on the pressure $p$ within the sealed-off borehole section as well as the pressure distribution $p_a(u)$ along the fracture. Zero fluid penetration into existing cracks yields a stress intensity factor of

$$K_1(p) = pR^{1/2}h_0(b).$$  \hspace{1cm} (6.24)

The dimensionless function $h_0$ was determined by Newman (1969) using the boundary collocation method. An analytical expression may be used to approximate the value of $h_0$, such that

$$h_0(b) = 1.3\frac{b - 1}{1 + b^{3/2}} + 7.8\frac{\sin\left[(b - 1)/2\right]}{2b^{5/2} - 1.7}$$  \hspace{1cm} (6.25)

which is shown in Fig. 6.14 in comparison with Newman's solution.

Similarly, the stress intensity due to a pressure acting within the crack may be expressed by

$$K_1(p_a) = pR^{1/2}h_a(b).$$  \hspace{1cm} (6.26)

![Figure 6.14. Dimensionless stress intensity functions $h_0(1 + a/R)$ and $h_a(1 + a/R)$ for a crack of half-length $a$ originating at the boundary of a pressurized circular hole. $h_0$ for a dry crack, $h_a$ for fluid-filled cracks with various pressure gradients. 1. Equation (6.25); 2. Newman (1969); 3. $p_a(x) = p$; 4. $p_a(x) = 0.8p$; 5. linear pressure distribution; 6. quadratic pressure distribution. Curves 1–3 show $h_0$, curves 4–6 show $h_a$ pressure within the hole, $p_a(x)$ pressure distribution along the crack.](image-url)
where \( p_a = p_a(x,0) \) describes the pressure distribution within the crack and \( p \) is the pressure within the borehole. Considering various cases of pressure gradients along the crack, the dimensionless stress intensity functions \( h_a \) are obtained by integration of Equation (6.13) as follows:

(a) constant pressure:

\[
p_a(x) = p
\]

\[
h_{a1} = (\pi b)^{1/2} \left( 1 - \frac{2}{\pi} \sin^{-1} \frac{1}{b} \right),
\]

(b) reduced constant pressure:

\[
p_a(x) = \nu p; \quad 0 \leq \nu \leq 1
\]

\[
h_{a2} = \nu h_{a1},
\]

(c) reciprocal pressure drop:

\[
p_a(x) = \begin{cases} -pR/x & \text{for} \quad -(R + a) \leq x \leq -R \\ pR/x & \text{for} \quad R \leq x \leq R + a \end{cases}
\]

\[
h_{a3} = 2(\pi b)^{-1/2} \ln \left[ b + (b^2 - 1)^{1/2} \right],
\]

(d) linear pressure drop:

\[
p_a(x) = \begin{cases} p(1 + R/a + x/a) & \text{for} \quad -(R + a) \leq x < -R \\ p(1 + R/a - x/a) & \text{for} \quad R \leq x \leq R + a \end{cases}
\]

\[
h_{a4} = \left( 1 + \frac{R}{a} \right)(\pi b)^{1/2} \left( 1 - \frac{2}{\pi} \sin^{-1} \frac{1}{b} \right) - 2 \left( \frac{b}{\pi} \right)^{1/2} (b + 1)^{1/2}(b - 1)^{-1/2},
\]

(e) quadratic pressure drop:

\[
p_a(x) = \begin{cases} p \left( \frac{a + R + x}{a} \right)^2 & \text{for} \quad -(R + a) \leq x \leq -R \\ p \left( \frac{a + R - x}{a} \right)^2 & \text{for} \quad R \leq x \leq R + a \end{cases}
\]

\[
h_{a5} = \left( \frac{b}{\pi} \right)^{1/2} (b - 1)^{-2} \left[ 3b^2 \left( \frac{\pi}{2} - \sin^{-1} \frac{1}{b} \right) + (1 - 4b)(b^2 - 1)^{1/2} \right].
\]

The dimensionless functions \( h_a \) are shown graphically in Fig. 6.14. The combined effect of pressure acting in the hole as well as within the crack is demonstrated in Fig. 6.15.
Figure 6.15. Dimensionless stress intensity function \( h = h_o + h_a \) for various pressure distributions in the hole crack system. 1. \( p_c(x) = p \); 2. \( p_c(x) = 0.75p \); 3. linear pressure drop; 4. quadratic pressure drop along the crack.

6.3.4. Superposition of Stress Intensity Factors

Superposition of the stress intensity functions for cases (i) to (iii) as suggested by Equation (6.13) yield the following relation for the critical borehole pressure at unstable crack extension:

\[
p_c = \frac{1}{h_o + h_a} \left( \frac{K_{ic}}{\sqrt{R}} + S_H f + S_h g \right)
\]  

(6.32)

where \( K_{ic} \) is the fracture toughness of the rock for pure mode I crack growth. If we compare this fracture mechanics solution with the classical fracture equation (e.g. Equation (6.1)) the term

\[
p_{co} = \frac{K_{ic}}{(h_o + h_a)\sqrt{R}}
\]

(6.33)

may be interpreted as the hydraulic fracturing tensile strength of the rock measured under zero external stresses. Equation (6.33) clearly indicates that the measured value of \( p_{co} \) for a given rock \( (K_{ic} \) constant) is size-dependent and decreases with increasing borehole diameter. Similarly, the coefficients of \( S_H \) and \( S_h \) may be defined as fracture coefficients as introduced by Equation (6.6):

\[
k_1 = \frac{g}{h_o + h_a}
\]

(6.34a)
6. HYDRAULIC FRACTURING STRESS MEASUREMENTS

Figure 6.16. Fracture coefficient \( k^* \) from Equation (6.36) as a function of normalized crack length \( 1 + a/R \) for fluid-filled cracks \( (\rho = \rho_a) \). 1. \( \varphi = 2.7 \); 2. \( \varphi = 2.5 \); 3. \( \varphi = 2.3 \, \text{g cm}^{-3} \).

\[
k_2 = \frac{f}{h_o + h_a}
\]  
(6.34b)

The functions reduce to values \( k_1 = 3 \) and \( k_2 = -1 \) for zero crack length which then correspond to the stress concentration factors of the classical hydrofracture equation for a crack-free rock. For the specific case of a lithostatic stress field the fracture coefficient (Equation (6.4)) is given by

\[
k = \frac{f + g}{h_o + h_a} = 1 - b^2 \frac{f}{h_o + h_a}
\]  
(6.35)

and the \textit{in situ} fracture gradient with respect to depth \( z \). \( k^* \) in Equation (6.5) may be estimated from

\[
k^* = g_o \left( \frac{f + g}{h_o + h_a} \sigma_z - \sigma_{\beta} \right)
\]  
(6.36)

where \( g_o \) is acceleration due to gravity. The variation of \( k^* \) as a function of crack length for rocks with different density values is given in Fig. 6.16 which may be used to estimate the length of pre-existing cracks from fracture gradients derived from \textit{in situ} hydrofracture experiments.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Dimension</th>
<th>Mini-cores Falkenberg granite</th>
<th>1 m³ block Epprechtstein granite</th>
<th>Deep boreholes Falkenberg granite depth 250 m</th>
</tr>
</thead>
<tbody>
<tr>
<td>borehole radius</td>
<td>$r, R$</td>
<td>mm</td>
<td>1.25</td>
<td>15</td>
<td>48/66</td>
</tr>
<tr>
<td>external field stresses</td>
<td>$p_m, S_{II}, S_h$</td>
<td>MPa</td>
<td>0–80 (Fig. 6.1)</td>
<td>$S_h = 2.0$</td>
<td>$S_h = 5.0$</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>$S_{II} \approx 5.0$</td>
<td>$S_{II} \approx 7.0$                       (Fig. 6.8)</td>
</tr>
<tr>
<td>breakdown pressure</td>
<td>$p_c, P_c$</td>
<td>MPa</td>
<td>16–100 (Fig. 6.1)</td>
<td>15.5 (Fig. 6.2)</td>
<td>10–17 (Figs 6.4, 6.6)</td>
</tr>
<tr>
<td>refracture pressure (fracture extension pressure)</td>
<td>$p_r, P_r$</td>
<td>MPa</td>
<td>—</td>
<td>$p_{r1} = 8.0$</td>
<td>8–10 (Fig. 6.7a)</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>$p_{r2} = 3.5$</td>
<td>(Fig. 2)</td>
</tr>
<tr>
<td>shut-in pressure</td>
<td>$p_{si}, P_{si}$</td>
<td>MPa</td>
<td>—</td>
<td>2.0</td>
<td>~5 (Fig. 6.7b)</td>
</tr>
<tr>
<td>hydrofracture tensile strength</td>
<td>$p_{co}, P_{co}$</td>
<td>MPa</td>
<td>$16.6 \pm 1.5$</td>
<td>~13.5</td>
<td>4–9</td>
</tr>
<tr>
<td>fracture coefficient</td>
<td>$k$</td>
<td>—</td>
<td>$1.04 \pm 0.04$</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>fracture toughness (Müller and Rummel, 1986)</td>
<td>$K_{lc}$</td>
<td>MNm$^{-3/2}$</td>
<td>$1.79 \pm 0.22$</td>
<td>$2.47 \pm 0.20$</td>
<td>$1.79 \pm 0.22$</td>
</tr>
</tbody>
</table>
6.4. APPLICATION OF FRACTURE MECHANICS TO EXPERIMENTAL RESULTS

The validity of the fracture mechanics model as developed above can be demonstrated for the experimental results obtained from hydrofracture tests in granites described earlier (minicores, 1 m² block, deep borehole tests at the Falkenberg granite test site). The relevant experimental data are summarized in Table 6.2.

First, application of Equation (6.33) allows us to estimate the intrinsic microcrack length of intact granite from the observed tensile strength values. The estimate yields

\[
\begin{align*}
a_{(ii)} &= 2 \text{ to } 3 \text{ mm} \quad \text{(mini-core, } p_{co} = 16.6 \text{ MPa)} \\
a_{(iii)} &= 4 \text{ mm} \quad \text{(1 m}^2 \text{ block, } p_{co} = 13.5 \text{ MPa)} \\
a_{(iii)} &= 7 \text{ to } 8 \text{ mm} \quad \text{(field test, } p_{co} = 7 \text{ MPa)}.
\end{align*}
\]

Such intrinsic crack lengths in otherwise intact (unjointed) granite are perfectly acceptable if one considers the coarse-grained matrix and the microscopic observation that grain boundaries act as potential microcracks under tensile loading. The observation that tensile strength values decrease from laboratory to field tests is obviously an effect of the increasing sample size which demonstrates that the probability of finding larger microcracks increases with the surface of the borehole wall exposed to the pressurizing fluid. Accepting these microcrack sizes for granite enables one to use Equation (6.35) to explain the small fracture coefficient \(k = 1.04\) derived from fracture tests on minicores at various confining pressures. This fracture coefficient should be compared to the value of \(k = 2\) as predicted by the classical theory which neglects the existence of microcracks. It should be mentioned that both estimations assumed that the fluid pressure within the borehole is the same as the fluid pressure within the microcrack \((p = p_a)\). The approximation can be improved if in addition one considers fluid pressure gradients due to viscous flow within the cracks. Using Equation (6.32) for the fracture test conducted in the Epprechtstein granite block subjected to \(S_H = 5 \text{ MPa and } S_h = 2 \text{ MPa}\) yields the fracture equation for initial breakdown (test F in Fig. 6.2).

\[
p_c = 13.6 - 0.28S_H + 1.21S_h
\]

(6.37)

which certainly is different to the classical fracture equation (Equation (6.1)). Inserting the values of the external field stresses the resulting initial crack extension pressure \(p_c\) is 14.6 MPa, a value sufficiently close to the observed breakdown pressure of 15.5 MPa. During the first injection the initial crack was extended to about 40 mm as indicated by the crack extension pressure of about 8 MPa in the second injection test. During
this phase the crack reached a length of 50 to 100 cm resulting in a critical pressure of about 3.5 MPa which was measured in the subsequent test (RF2 in Fig. 6.2). Again the estimations are based on the assumption that the borehole pressure also acted over the total length of the crack. Since in the test the pumping rate was extremely small (mm$^3$ min$^{-1}$) the assumption may be valid.

Finally, application of Equation (6.32) to the deep hydrofracture tests carried out at the Falkenberg granite test site yield the fracture relation for the initial breakdown

$$P_c = 6.7 - 0.3S_H + 1.35S_h$$

(6.38)

which again contradicts the classical concept of a crack-free rock. If we use the stress values $S_H = 7$ MPa, $S_h = 5$ MPa as derived from the field data and using Equation (6.3), the breakdown pressure is calculated as $P_c = 11$ MPa which agrees with values observed in the field tests. At present, further analysis to predict refracture pressure values, or to use them to speculate on the stress field data does not seem justified. Generally the tests have been conducted at rather high injection rates (10 l min$^{-1}$) so that we may have to consider a pressure gradient within the large but narrow induced fractures.

6.5. CONCLUDING COMMENTS

The fracture mechanics treatment of hydraulic fracturing presented above is only a first approach to the problem. The analysis is two-dimensional, it does not include fluid losses from the crack surface into the rock mass, assumes rock mass isotropy and neglects formation boundaries. Three-dimensional crack growth was originally considered by Sack (1946) for the special case of penny-shaped cracks. Fluid losses for a radially expanding fracture were treated by Abé et al. (1976). Anisotropy and crack extension in directions different to $S_H$ were discussed by Abou-Sayed et al. (1978). Formation boundaries for crack growth were considered by van Eekelen (1982) and many other authors who have considered fracture growth in oil and gas industry.

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REFERENCES